Examples of the Index

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The goal is to compute the topological and analytic index for some simple compact manifolds, the point, the circle. There are two perspectives we can take on *what the index is*, one is that it is the map on K theory that is introduced, the other is perhaps more true. The machinery we have seen takes an operator and gives you a class in K theory, then argues that this map is invertible on the level of the Fredholm index, thus we can apply the Fredholm index to the fiber and get an integer. The claim is that this is the same as the topological index as defined.

$$\begin{tikzcd}
\mathrm{Op}(E,F;X) \arrow[r, bend left, "\mathrm{symb}"] \arrow[dr, bend right, "\mathrm{fred-ind}"'] & K(TX) \arrow[l, bend left, "\mathrm{symb}^{-1}"] \arrow[d, "\mathrm{t-ind}"] \\
& \mathbb{Z}
\end{tikzcd}$$

The more intuitive way of thinking of this process is really that if we start with an operator there are two ways of getting an integer, one is the Fredholm index and the other is via the symbol and the topological index.

Remark. We will return to more non-trivial examples when seeing the applications of this theorem.

1 The Point

Topological. Now we look at the point as a compact 0 dimensional manifold. Its (co)tangent bundle is again the point and therefore the K theory of the cotangent bundle is the K theory of the point. In this case the topological index, from its definition, must be an isomorphisms from $\mathbb{Z}$ to $\mathbb{Z}$ and hence multiplication by ± 1, we normalise it to be $+1$.

Analytic one. From the point of view of the analytic index we want to see that a generator for the K theory over the point, the trivial line bundle, gives some operator class whose Fredholm index is also 1. Because we are over the point our linear operators are between finite dimensional spaces and it simplifies greatly. We are looking for something of the form $0 \to \pi^* E_0 \to \pi^* E_1 \to 0$ where these are complex bundles over the point being pulled back by the projection from the tangent bundle over the point (also the point). But because we want them to map to the line bundle in K theory we can just take them to be the line bundle over the point and the zero dimensional vector bundle over the point respectively. From here our choice about differential operators are fixed.

That is the logic, here is the cleaner version. Consider the following bundles over the point: $\mathbb{R}, * \to *$, that is the line bundle which we denote [1] and the rank 0 vector bundle denoted [0]. Then a differential operator is a map

$$\Delta : \Gamma([1]) = \mathbb{R} \to \Gamma([0]) = *$$

Of which there is clearly only one. Moreover the Fredholm index of this unique operator is given by

$$\text{fred} - \text{ind}(\Delta) = \dim\ker\Delta - \dim\text{Im}\Delta = 1 - 0 = 1.$$

Finally we can find the class in the K theory for this operator. These two bundles by the construction of the pullback just pullback to again the trivial one and zero dimensional bundles over the tangent bundle over the point, which is the point; they are pulled back along the identity, they are themselves. Thus there is a sequence

$$0 \to [1] \xrightarrow{\text{sym}\Delta} [0] \to 0$$

Which clearly maps to [1] in K theory by taking the alternating sum.

Analytic two. From the other perspective we might want to start with two bundles over the point, or just two vector spaces, and consider linear operators between them. Then given such an operator take its Fredholm index and on the other find the class in K theory of the symbol and then compute the topological index. This is linear algebra. Lets do it.

If we have two finite dimensional vector bundles over the point E, F or dimension n, m respectively then their space of sections is just isomorphic to the fiber and so the space of sections is again just the fiber, thus a linear map between the sections of these bundles is a linear map

$$\mathbb{C}^n \to \mathbb{C}^m$$

The Fredholm index of such a map is then

$$\dim\ker - \dim\text{coker}$$

notice that because domain = ker $\oplus$ Im and codomain =Im $\oplus$ coker we get that

$$n - m = \dim\text{domain} - \dim\text{codomain} = \dim(\ker\oplus\text{Im}) - \dim(\text{Im}\oplus\text{coker}) = \dim\ker - \dim\text{coker}$$

which is the Fredholm index.

Given such a linear map its class in K theory is given, by a similar argument as above, by $[n] - [m]$, and hence so is its topological index.

Remark. We are using the complex dimension of the subspaces for the Fredholm index. It shouldnt matter either way, just as long as we keep track.

Remark. We are really using quite strongly that the tangent bundle is also compact, otherwise finding the class in K theory wouldnt be as simple as taking the difference.

2 The Circle

2.1 The K theory of spheres

Lets first discuss vector bundles over spheres in general. Because the sphere is two discs glued along their boundary $S^n = D^n \cup_{S^{n-1}} D^n$, a vector bundle over the sphere will have two local trivialisatoins on these discs. If given two vector bundles (of the same rank) over the disc D^n then to know how to peice them back together we need an identification of the fibers of the bundles on the boundary. If we fix two trivialisations one for each of the bundles on the discs, then this is a choice of where to send the basis from the fiber of the first disc in a bijective way to the fiber of the second disc (at the same point on the boundary). But because the fibers are both $\mathbb{R}^k$ as vector spaces this is just an automorphism of $\mathbb{R}^k$, i.e. an element of $\mathrm{GL}_k(\mathbb{R})$. Therefore we claim that

$$\mathrm{Vect}_k^{\mathbb{R}}(S^n) \cong [S^{n-1}, \mathrm{GL}_k(\mathbb{R})]$$

Note that we are making several claims here: That these assignments from the boundary of the discs to automorphisms of the fibers are sufficient to construct bundles (intuitive) (you require that they vary smoothly in $\mathrm{GL}_k(\mathbb{R})$ to ensure that they glue together), that all bundles are constructed this way (also intuitive) and that bundles are isomorphic iff their so called "clutching functions" are homotopic. reference? Huesmoller/Hatcher?

For complex bundles it is the same thing. By the standard argument $\mathrm{GL}_n(\mathbb{C})$ deformation retracts onto its maximal compact subgroup U, and so we can simplify it to

$$\mathrm{Vect}_k^{\mathbb{C}}(S^n) \cong [S^{n-1}, \mathrm{U}(k)]$$

These groups are not in general friendly. Luckily we do not need to compute them to compute K theory as K theory classes are in bijection with only *stable* vector bundles (this is in Atiyahs book or my other notes). We can see directly from our presentation that taking the colimit over all vector bundles therefore will give us the stable vector bundles, hence

$$\mathrm{Vect}_{\mathrm{stab}}^{\mathbb{C}}(S^n) = \tilde{K}(S^n) = \mathrm{colim}_k[S^{n-1}, \mathrm{U}(k)] = [S^{n-1}, \mathrm{U}] = \pi_{n-1}U$$

Here we are using the fact that we can take the colimit inside, this is ok because S^{n-1} is compact, I believe. These are just the homotopy groups of the infinite unitary group, which are well known by Bott periodicity to be 0 in even degrees and $\mathbb{Z}$ in odd degrees.

Note that we know from the spectra presentation of (reduced) K theory that a priori $\tilde{K}(X) = [X, BU]$ (unreduced has a product with $\mathbb{Z}$). In the special case of the spheres we can deduce from this the formula above,

$$\begin{aligned}\tilde{K}(S^n) &= [S^n, BU] \\ &= [\Sigma S^{n-1}, BU] \\ &= [S^{n-1}, \Omega BU] \\ &= [S^{n-1}, U]\end{aligned}$$

2.2 The K theory of the tangent bundle

The tangent bundle and cotangent bundle are diffeomorphic and so at least from the perspective of K theory they will have isomorphic groups. The tangent bundle on the circle is just the trivial line bundle. Therefore the Thom space is easily seen to be the pinched torus, for example by thinking of it as the disk bundle mod the sphere bundle. This is topologically the 2-sphere with any two distinct points identified.

So whats its K theory? A vector bundle is just a vector bundle on the sphere with a homotopy class of a map from the point into $\mathrm{GL}_n(\mathbb{C})$, in other words, the same as a vector bundle on the sphere, because $\mathrm{GL}_n(\mathbb{C})$ is connected. In otherwords any vector bundle on the sphere can glue the fibers at two distinct points by the local triviality condition, and up to homotopy there is only one way to do this. Moreover all bundles will arise in this way (intuitive). Alternatively the pinched torus is homotopic to the wedge $S^1 \wedge S^2$ and K theory is additive on wedges, $\tilde{K}(S^1 \wedge S^2) = \tilde{K}(S^1) \oplus \tilde{K}(S^2) = 0 \oplus \mathbb{Z}$. So $K(T^\vee S^1) \cong \tilde{K}(S^2) = [S^1, U] = \mathbb{Z}$.

We would like to find a generator in this K group. The generator is given by a generator in $\tilde{K}(S^2)$ which by the calculation of this K group is the generator for $\pi_1 U$. Now for $n \le N$ the canonical inclusion $U(n) \to U(N)$ is $2n$-connected, that is an isomorphism on π_i for $i \leq 2n$. In this precise sense the fundamental group of $U(N)$ therefore comes from the fundamental group of $U(1) \cong S^1$ which is just $\mathbb{Z}$ again, and moreover has a canonical generator given by the identity. Hence if we think of the K theory as equivalence classes of clutching functions under group completion, then this is the class given by the identity $S^1 \to U(1)$.

According to the nlab the generator of the Bott periodicity is the tautological bundle, given by thinking of S^2 as $\mathbb{C}P^1$. Note that there they are unreduced and so they write that the Bott element is $[1] - [T] \in K(S^2) = \mathbb{Z} \oplus \mathbb{Z}$, however for us we will be reduced and so we will think of it as just the tautological bundle as $[1] = (1, 0)$.

2.3 The Index

Now we know that the topological and analytic index are group homomorphisms from $\mathbb{Z} \to \mathbb{Z}$ and are therefore given by multiplication by an integer (the choice of where to send 1). Therefore it is sufficient to determine the analytic and topological index of the generator in K theory. We determined this generator above as the identity clutiching function or the tautological line bundle on $\mathbb{C}P^1$.

We can fix an embedding therefore of $S^1 \hookrightarrow \mathbb{R}^2$. Since $T\mathbb{R}^2 = \mathbb{R}^4$ whose Thom space is simply S^4 we get that

$$\tilde{K}(S^2) \to \tilde{K}(S^4) \to K(*)$$

All of these groups are $\mathbb{Z}$ and because the last map is an isomorphism it is given by multiplication by ± 1. Hence up to a sign the topological index is determined as the pushforward of the embedding $S^1 \to \mathbb{R}^2$. So take a tubular neighbourhood around S^1 in $\mathbb{R}^2$, which will be an anulus $S^1 \times I$. Then we get maps on the tangent bundles

$$\begin{tikzcd}
TS^1 \arrow[r] \arrow[d, equal] & T(S^1 \times I) \arrow[r] \arrow[d, equal] & T(\mathbb{R}^2) \arrow[d, equal] \\
S^1 \times \mathbb{R} \arrow[r] & S^1 \times I \times \mathbb{R} \arrow[r] & \mathbb{R}^4
\end{tikzcd}$$

Then on K theory we have

$$\tilde{K}(S^2) \to \tilde{K}(\mathrm{Th}(S^1 \times I \times \mathbb{R})) \to \tilde{K}(S^4)$$

where the first map is multiplication by the Thom class in $\tilde{K}(\mathrm{Th}(S^1 \times I \times \mathbb{R}))$ and the second is induced by the LES of the pair. The middle K group is again just $\tilde{K}(S^2)$ as attaching the ends of the macaroni and contracting the I (its a homotopy invariant) again gives us the pinched torus (alternitively we see that by the Thom isomorphism it must be the same as the K theory of the tangent bundle over the cirle). Since the Thom class is a generator it can be chosen to be ± 1, and so the first Thom map is just multiplication by ± 1. The only thing then to calculate is the map

$$\tilde{K}(\mathrm{Th}(S^1 \times I \times \mathbb{R})) \to \tilde{K}(S^4)$$

$$\mathbb{Z} \to \mathbb{Z}$$

This map arrises in the follwing way, using relative and compactly supported K theory

$$\begin{tikzcd}
\tilde{K}_c(TY, TY - TN) \arrow[r] \arrow[d, phantom, "\cong"] & \tilde{K}_c(TY, \emptyset) \arrow[d, phantom, "\cong"] \\
\tilde{K}_c(TY - (TY - TN)) \arrow[d, phantom, "\cong"] & \tilde{K}_c(TY - \emptyset) \arrow[d, phantom, "\cong"] \\
\tilde{K}_c(TN) \arrow[r, dashed] & \tilde{K}_c(TY)
\end{tikzcd}$$

and the top map is given by functoriality and the inclusion of the pair $(TY, \emptyset) \hookrightarrow (TY, TY - TN)$. Another way to see what this map is as applying K to the quotient map, the so called Pontryagin-Thom collapse map,

$$X^+ \to X^+/(X^+ - U) \cong U^+$$

where X is a locally compact space and U is an open subset. Because the normal bundle is a trivial rank 2 bundle (its Thom space is given by smashing the base space with S^2), in our case this looks like

$$S^4 \to \mathrm{Th}(TN) = S^2 \wedge S^1$$

So we need to examine what pulling back the following diagram does

$$\begin{tikzcd} E \times_c S^4 \arrow[r, dashed] \arrow[d, dashed] & E \arrow[d] \\ S^4 \arrow[r, "c"] & S^2 \wedge S^1 \end{tikzcd}$$

in particular where E is the Bott element over S^2, that is the Tautological bundle (we want to see that it maps to a generator). The claim is that the collapse map is degree one and therefore by definition is given by multiplication by ± 1 and hence is an isomorphism.

Analytic one. If we take the generator of the K theory that we have previously found, namely the tautological bundle, then can be find an operator to which it corresponds and then compute its Fredholm index?

Analytic two. Here we want to take an opperator and compute its index in two different ways. Lets do something simple. Consider $L \to S^1$ the trivial line bundle. Then the smooth global sections of this bundle are smooth maps $S^1 \to \mathbb{C}$ hence an operator between the global sections is a linear function

$$C^\infty(S^1) \to C^\infty(S^1)$$

The simplest differential operator we can think of in this case is just differentiation $i\partial/\partial x$, where the i factor is to ensure self adjointness. Because this function is self adjoint and there is an isomorphism between $\mathrm{coker} f$ and $\ker f^*$ its adjoint it is immediate that the index of such a self adjoint elliptic operator (we will see ellipticity later) must be zero.

Now we want to compute the (principle) symbol of this operator. Clearly in local coordinates it is itself, moreover it is clearly order 1. When we perform the substitution $\partial/\partial x \to i\xi$ the i factors multiply and so the symbol is just (independent of x)

$$\sigma\left(i\frac{\partial}{\partial x}\right)(\xi) = -\xi.$$

In other words the coefficient function (in the notation of my notes on the symbol) is given by $p(x) = i$ and so as a function on bundles it is just fiberwise multiplication by i. Thus we are looking at the following element in K theory

$$0 \to L \xrightarrow{\times i} L \to 0$$

we want to now take the topological index of this bundle. Multiplication by a constant is homotopic to multiplication by 1, and therefore this is just (in K theory) the identity sequence. Thus we just get the formal difference of $L - L = 0$ and since our map is a group homomorphism this is itself sent to zero.

Another way to see this is that this sequence above, because multiplication by a non-zero number is a bijection, is always exact. Therefore the support of the sequence is the empty set, and so we by definition mod out by it. Alternitively we consider it as an element of the relative K theory $K(TS^1, \mathrm{supp})$, which is given by identifying fibers on the support, but the support is empty and so we dont have to worry about the gluing process and just get the formal difference.